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Learners' Functional Understandings of Proof (LFUP) in Mathematics: A Qualitative Approach

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Abstract

The purpose of this study was to present a revision and validation of the Learners' Functional Understandings of Proof (LFUP) scale in mathematics using data collected from Grade 11 learners (n = 87) in a high school in South Africa. The LFUP scale was linked to the five-factor model (verification, explanation, communication, discovery, and systematisation) whose items were derived from existing literature on proof functions. Unlike the previous version of the scale, the new scale being validated here blends Likert-scale and constructed-response items to evaluate learners' conceptions of the essence of the functions of an aspect central to mathematical knowledge development: proof. It is my contention that the LFUP instrument can be used as either a summative or a formative assessment tool, given the argument that learners often require motivation as to why they are required to write proofs. In short, this study provided an instrument to introduce learners to the concept of mathematical proof. Multiple regression analysis revealed that all five LFUP tenets correlated significantly with the total sum of all the functions of proof taken together. In the qualitative analysis, a substantial number of learners (52%) were found to hold hybrid beliefs about the functions of proof in mathematics.

Introduction

Research studies have shown that learners experience substantial difficulties with Euclidean proof (de Villiers, 2012; Hanna, de Villiers, Arzare, Dreyfus, Durand-Guerrier, Jahnke, Lin, Selden, Tall, & Yevdokimov, 2009; Harel & Sowder, 2007; Mudaly, 2007). Despite efforts to teach proof and improve learners' performance in Euclidean geometry, few American learners finish high school able to formulate conjectures and construct mathematical proofs (Hadas, Hershkowitz, & Schwarz, 2000). Given that the 'failure to teach proofs seems to be universal' (Hadas, Hershkowitz, & Schwarz, 2000, p. 128), functional understanding of proof and argumentation, activities Edwards (1997) refers to as the "territory before proof", need to be part of the mathematical activities that precede and support the development of proofs. Along this line, Marrades and Gutiérrez (2000) argue that it is vitally important for both teachers and researchers in the area of proof to know learners' conceptions of functions of mathematical proof in order to understand their attempts to solve proof problems. Why is proof key in mathematics?

The construction of proofs has always been regarded as a defining activity within the mathematics discipline (de Villiers & Heideman, 2014; Lockhart, 2002; Watson, 2008). One primary reason is that proof is an essential tool for fostering mathematical understanding (Ball, Hoyles, Jahnke, & Movshovitz-Hadar, 2002). Yet, inconsistent with the practices of research mathematicians, the focus of high school mathematics has often been on form and established results to pass examinations over the activities that are a precursor to the construction of proofs, for example, understanding the functions of proof and argumentation. Perhaps more importantly, unless learners understand the purpose in studying proofs beyond the goal of preparing for the next mathematics class or test, they are likely to ask the age-old question, *Why do we need to learn this*?

The general motivation for this study came from the need to measure learners' understanding of the functions of proof in mathematics since lack thereof contributes to difficulties with meaningful construction of proofs (de Villiers, 1990; Healy & Hoyles, 1999). According to the van Hiele (1986) theory, functional understanding of proof is one of the aspects that determine learners' ability to construct a deductive proof. Mathematical proof performs various functions in mathematics including verification, explanation, communication, discovery, systematisation, and intellectual challenge for the author of the proof. Although these functions are enshrined in mathematics education policy documents (Common Core State Standards Initiative [CCSSI], 2010; Department of Basic Education [DBE], 2011; National Council of Teachers of Mathematics [NCTM], 2000), learners' knowledge of these functions is not explicitly assessed and thus not measured. Brook and Stainton's (2001) definition of "knowledge" was adopted in this study on the basis that it was plausible, commonly used, and provided by philosophers (Ball, Hoyles, Jahnke, & Movshovitz-Hadar, 2002). They argued that for a proposition to be regarded as knowledge, it must be held as true by the mathematical community and the learner must have justification for believing it.

The Trends in International Mathematics and Science Study (TIMSS) 2011 report, which contains vital information on key factors that can impact the teaching and learning of mathematics, revealed that even learners in countries with high national mathematics performance struggle with Euclidean geometry. These data on learner achievement trends suggest that learners have particular difficulty with proof compared with other concepts within the geometry area. The difficulty to prove contributes to the distortion of mathematics as a discipline whose rules and procedures are supposed to be memorised to pass examinations and tests. This unfortunate perception has unintended consequences; the decline in university learners' enrolment in pure mathematics courses and therefore the reduction in the potential pool of future mathematicians. This reduction, in turn, limits the potential of scientific discoveries and technological innovations that could assist in ensuring food security and combating the devastating effects of climate given that mathematics is the queen of the sciences. Important not to overlook is that as fewer learners take mathematics in high school, fewer mathematics teachers are available. It is simply a vicious cycle.

Returning to the concept of proof, memorisation of proofs suggests that most learners cannot do proof for meaning. Thus, gaining insight into the challenges that learners have with proof and finding ways to improve its learning is crucial for enhancing struggling learners' success in proof and beyond. Why is meaningful learning of proof so difficult? Various reasons have been canvassed. In our daily lives we frequently encounter or use the term "proof". Although mathematicians are accustomed to think of "proof" as an unambiguous term (Epstein & Levy, 1995), it has a multiplicity of meanings to the extent that its meaning is still unclear in school mathematics (Stylianides, 2007). Along this line, Cabassut, Conner, Ersoz, Furinghetti, Jahnke, and Morselli (2012) point out that whereas mathematicians are convinced that, in practice, they know precisely what a proof is, there exist no easy explanations of what proof is that teachers can provide to their learners. The multiple definitions of proof contribute to the difficulty that learners experience in their learning of the concept. According to a widely disseminated definition of proof provided by the Principles and Standards for School Mathematics (National Council of Teachers of Mathematics [NCTM], 2000), proof pertains to the process in which conclusions are derived from axioms in a finite sequence of logical steps.

Further, Tall (1989) points out that the term "proof" means many different things to learners such that interpretation of its meaning may be different from that of the teacher, just as one teacher's interpretation may differ from another's. However, since the term "proof" has been used differently in many situations, in an academic discipline like mathematics education its exact meaning would seem to be important (Reid & Knipping, 2010). Similarly, Epp (2003) points out that mathematical language is required to be unambiguous. CadwalladerOlsker (2011) avers that this difficulty is further compounded by the fact that proof performs several different functions in mathematics and may be written for a specific audience.

In this study, a proof is viewed as an argument based on accepted truths for or against a mathematical claim (conjecture). The term "argument" is used to denote a connected sequence of statements generated from the axiomatic method. The term "axiomatic method" means a method of organising a theory (theorem) by beginning 'with the list of undefinable terms and unprovable axioms, including those terms from which the statements of the theory (theorems) should be deduced according to the rules of formal logic' (Demidov, 1980, p. 215). In keeping with de Villiers' (2012) caution, I did not define proof in terms of its verification or any of its multiple functions, to avoid elevating a particular function as more important than the others. For instance, Griffiths' (2000) idea of proof as 'a formal and logical line of reasoning that begins with a set of axioms and moves through logical steps to a conclusion' (p. 2) reflects the systematisation function of proof as it mentions that proof begins with assumptions and logically connecting them to reach a conclusion.

Clearly, the question "What is a mathematical proof?" is difficult to answer despite the extensive literature on proof. However, Stylianides (2007) provides an apt definition of proof, emphasising argumentation:

> *Proof is a mathematical argument, a connected sequence of assertions against a mathematical claim, with the following characteristics:*
> *1. It uses statements accepted by the classroom community (set of accepted arguments) that are true and available without further justification;*
> *2. It employs forms of reasoning (modes of argumentation) that are valid known to, or within the conceptual reach of, the classroom community;*
> *3. It is communicated with forms of expression (modes of argument representation) that are appropriate and known to, or within the conceptual reach of, the room community. (p. 291)*

Having defined proof in this manner, learners' understanding of the functions of proof was understood at three broad and distinct levels: naïve; hybrid; and informed. That is, learners who understand that proof has no functions other than verification are classified as holding "naïve" views about its functions while those who understand the other functions that proof performs in mathematics were labelled as holding "informed" views and thus assumed as being able to formulate and prove propositions. The intermediate level at which the understanding of the functions of proof included both naïve and informed understanding of the functions of proof was labelled as "hybrid".

According to the DBE (2018) diagnosis, there has been little progress in addressing the persistent underachievement of learners in this domain of mathematics. Various sources of this difficulty have been identified. Easdown (2012) suggests that this difficulty manifested itself in three ways: appreciating why proofs are important; the tension between verification and understanding; and, proof construction. That is, the multiplicity of meanings ascribed to the term "proof" contaminates learners' ability to distinguish between its everyday use and technical meaning. In support of Easdown (2012), de Villiers (1990) concludes that on the basis of extensive interviews with learners, most learners' difficulty with proof seem to 'not lie so much with poor instrumental proficiency nor inadequate relational or logical understanding as in poor understanding of the usefulness or function thereof' (p. 11).

Theoretical Framework

Further validation of the LFUP instrument was framed and guided by the following five theoretical constructs on the functions of proof in mathematics. I preferred to make a distinction between proposition and statement. By proposition and statements here I respectively meant a conjecture whose actual proof is under construction and an axiom, definition, major concept, or theorem used in the construction of a proof. In the two-column proof format, the latter (statements) are the items on the left hand side. The next section frames this study on de Villiers (1990) model:

- *Verification*. Proof as a means to verify the truth of a proposition refers to viewing it as a tool to establish certainty of a conjecture (Stylianides & Stylianides, 2018), and thus to confirm one's intuition (Schoenfeld, 1992).
- *Explanation.* Proof as a means to explain entails the provision of insight into, and recording how, a proposition comes to be true (Herbst & Miyakawa, 2008) using well-known and well-understood properties of the mathematical objects involved (Hanna, et al., 2009).
- *Communication*. Proof as a means to communicate mathematical knowledge concerns sense making, learning of mathematical language, and "transmission" of socially constructed knowledge publicly (thus creating a forum for critical debate) – both in discourse and in writing – that is acceptable to the mathematical community (de Villiers & Heideman, 2014).
- *Discovery*. The discovery function of proof related to the generation of new results (theorems) in which new ideas, concepts, and methods emerge (Rav, 1999). Similar to Stylianides (2009), the phrase "new results" is used to describe proof knowledge that learners added to their existing knowledge base as a result of constructing a proof.
- *Systematisation*. Proof as a means to systematise mathematical knowledge refers to the organisation of results previously thought to be unrelated into the existing deductive system of axioms, major concepts (definitions), and theorems (de Villiers, 1990).

Aims and Research Questions

The aim of this study was to establish the validity of an existing instrument designed to measure Learners' Functional Understandings of Proof (LFUP) in mathematics by blending qualitative measures with quantitative aspects. Although the LFUP scale has been found to be a valid and meaningful instrument elsewhere (see, Shongwe & Mudaly, 2017), further validation was conducted to improve its use not only as either a summative or formative assessment tool, but also to track competence in the learning and teaching of mathematics. The endeavour to construct such a scale found expression in the standards and curriculum documents. For instance, both the *Principles and Standards* in the National Council of Teachers of Mathematics (NCTM) (2000) and the South African *Curriculum and Assessment Policy Statement* (CAPS) (Department of Basic Education [DBE], 2011) propose greater emphasis on the making and testing of conjectures, the formulation of counterexamples and the construction of deductive arguments.

Effective endeavours aimed at developing learners' informed views of the functions of mathematics require a clearer picture of the current baseline views of these functions. Thus, the main research question that guided this study in gaining a more complete picture of LFUP was 'What are learners' functional understandings of proof in mathematics?' The following sub-questions elucidated this research question and facilitated data analysis:

- What is the proportion of variance in learners' functional understandings of proof that can be explained by the five functions of proof?
- Which of these five factors is the best predictor of learners' functional understandings of proof in mathematics?
- How do learners view the functions of proof in mathematics?

Methods

Participants and Procedures

The present study began after approval by the Ethics in Research Committee of the University of KwaZulu-Natal (Protocol number: 2/4/8/1126) as well as the KwaZulu-Natal Department of Education (Protocol number: HSS/0437/016M) was received. A nonrandom, purposive sample of 87 participants was selected from two classes in one of ten Dinaledi high schools[1] in the Pinetown Education District in KwaZulu-Natal (South Africa) to establish validity for the LFUP instrument. The mean age of the participants was 17.35, with a range between 16 and 20 years, 62% of which were female. The school was located in a township and was not part of the main study. The demographic information collected included age, home language, and gender. The total time allocated for the administration of the task was 60 minutes. Piloting of the task also provided an opportunity to pose questions to participants regarding their understanding of the wording, criteria, and instructions, and to evaluate their appropriateness for different subgroups of learners. A pilot study helped to identify learners' reading comprehension of the questionnaire items because this 'may be affected by difficulty of the text, the vocabulary words used in the text, and the reader's familiarity with the subject matter, among other factors' (Nicolas & Emata, 2018, p. 41).

Development of the LFUP Scale

My supervisor and I started investigating the validity of the LFUP scale as part of my doctoral studies in 2017. I then made a commitment to embark on a journey to develop a long-term research agenda to gather evidence using different samples to establish whether the claims of validity were reasonable. Initially, the literature on functions of proof was reviewed and a 25-item questionnaire was constructed from a pool of 30 items. A convenience sample of 10 learners from the target population of Grade 11 learners, together with two fellow doctoral students at the time, were invited to comment on the face and content validity of the questionnaire. In addition, three international experts in proof and its functions were requested to comment on the items that constituted the initial version of Test. A *t*-test was used in the initial study.

Instrumentation

The LFUP instrument consisted of 25 questions in which Likert scale items with five subscales and follow-up constructed-response items were embedded within their respective subscales. Likert response categories ranged from *strongly disagree* (1) to *undecided* (3) to *strongly agree* (5) thus indicating the extent to which statements reflected learners' opinions on the functions of proof in mathematics. A sample of questionnaire items on the LFUP questionnaire are depicted in Table 1. The use of these open-ended items was meant to 'capture idiosyncratic

[1] In the quest to increase the participation and performance of historically disadvantaged learners in mathematics and physical sciences, the Department of Basic Education established the Dinaledi School Project, in 2001 (Department of Basic Education [DBE], 2009).

differences' (McMillan & Schumacher, 2010, p. 198), eliminate the effect of guessing, and facilitate subsequent categorisation of data for analysis.

Table 1: Sample items theoretically clustered under the "systematisation" function of proof

Function	Code	Description
Systematisation	s1	Proving doesn't require deciding which statements to be chosen as true.
	s2	Proving involves reasoning & argumentation that's different from rest of maths.
	s3	Proving may lead to a replacement of statements that could be used in later proofs.
	s4	Proof brings together & connects maths results.
	s5	Proving may lead to addition of new statements to be used in later proofs.
With examples, explain how the theorems you are asked to prove were dependent OR independent of one another. ………………………………………………………………………………………… ………………………………………………………………………………………… …………………………………………………………………………………………		

The constructed-response items were assessed and classified into three categories: "adequate", "hybrid", or "inadequate", a categorisation that provided the basis for interpreting the data. Learners' constructed responses were assigned the code "adequate" category if they reflected notions of functions of proof which were deemed as consistent with current literature on proof, "inadequate" if they reflected views which represented common learners' naïve beliefs about the functions of proof that are not consistent with current literature on proof, and "hybrid" if they integrated elements of both adequate and inadequate views.

Results

Determination of the scale's reliability took two forms: internal consistency and item-total statistics. The high positive correlations were an indication of reliability of the LFUP questionnaire. By "item-total", it is meant the correlation between each item and the overall score of the scale (McDowell, 2006). An examination of the item-total correlations (Table 2) indicates that all items in each dimension contributed to the consistency of scores with item-total correlations higher than .51 thus exceeding the acceptable cutoff value of .30 (Tabachnick & Fidell, 2013).

The Cronbach's alphas of each subscale or factor on the LFUP instrument confirmed our hypothesis that the functions of proof are independent and therefore not interrelated. Specifically, the alpha values for each subscale were found to be as follows: Verification (α = .60); Explanation (α = .73); Communication (α = .69); Discovery (α = 0.74); and Systematisation (α = .78). Given that the alpha values approximated 1.00 across the five factors suggests that LFUP can be used as a reliable assessment and diagnostic tool in instructional practices and research.

Table 2: Item-total statistics for independent variables

Independent variables	Scale Mean if Item Deleted	Scale Variance if Item Deleted	Corrected Item-Total Correlation	Cronbach's Alpha if Item Deleted
AV	11.8588	6.218	.630	.766
AE	12.2824	6.324	.618	.770
AC	12.1882	6.321	.657	.757
AD	12.0588	6.699	.601	.775
AS	12.3176	7.719	.508	.802

AV-average of verification; AE-average of explanation; AC-average of communication; AD-average of discovery; AS-average of systematisation

Quantitative Analysis

Quantitative analysis of data consisted of two components. First, descriptive statistics and multiple regression were performed for the Likert scale items of the LFUP instrument. Statistical Package for the Social Sciences (*SPSS) Version 16* was used to perform these analyses. The data were screened for normality, outliers, and multicollinearity. Learners' responses were coded as 5 if they reflected the most informed view of functions of proof and as 1 for the naïve view. Mean scores for each of the five functions of proof and the overall LFUP instrument were determined. Thus, the five subscales (with 5 as maximum value) were correlated with the total score (with 35 as the maximum value) with a larger score demonstrating informed functional understandings of proof.

Sample characteristics

In order to ensure that inferences we made in the tests were valid, the sample data were drawn from a normally distributed population. A multiple of sources for evidence of normality were used. For example, as shown in Figure 1, a visual inspection of the histogram showed that the scores on learners' functional understandings of proof were approximately normally distributed.

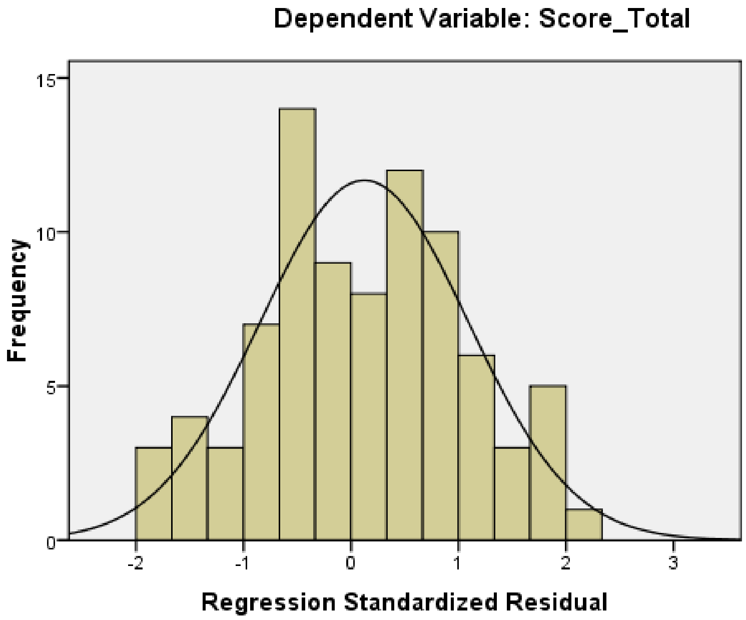

Figure 1: Histogram showing relative normal distribution of LFUP scores

As explained in Figure 2, the dots on the Normal P-P plot generally followed the diagonal line thus showing that the LFUP scores were approximately normally distributed with acceptable absolute values of skewness of .374 (SE = .365) and kurtosis of .584 (SE = .717), respectively (Doane & Seward, 2011).

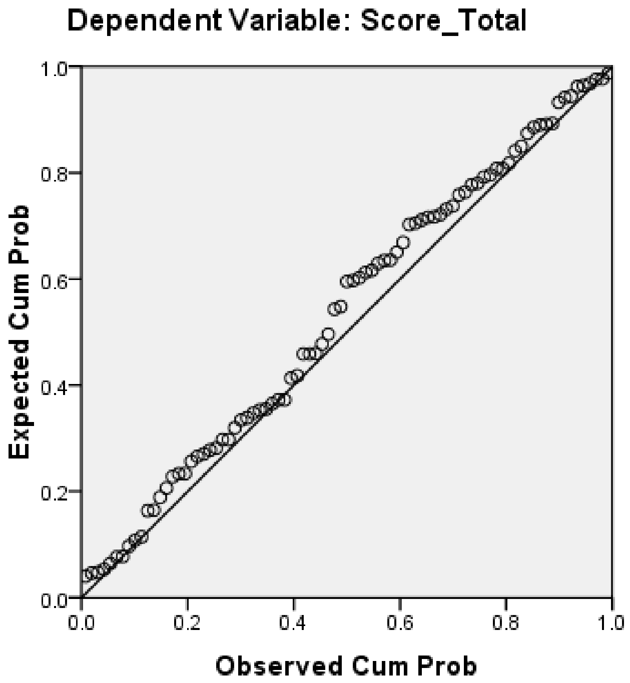

Figure 2: The Normal P-P plot for assessing normal distribution of data

The proportion of learners' functional understandings explained by the five subscales

Multiple linear regression was performed to investigate multicollinearity so as not to make interpretation of results difficult. That is, we needed to determine which of the functions were highly correlated with each other because if this were the case, it would be difficult to identify the function which best predicted learners' functional understandings of proof. Correlations of .80 or higher between subscales were indicative of multicollinearity (Wilson & MacLean, 2011). In this case, all the correlations between the subscales were significant in that they were above the threshold of .30 (Tabachnick & Fidell, 2013) and below the limit of .80 and therefore indicating that there likely was not a problem using any of two subscales correlated. More precisely, each of the variance inflation factors (VIF) were less than 10 (Myers, 1990) thus enabling the retention of all subscales. Further support for the absence of multicollinearity was found in the fact that the average VIF value was not substantially greater than 1 (Myers, 1990). As depicted in Table 4, correlation coefficients (r) for each of the five subscales (predictor variables) with the functional understandings of proof scores (criterion variable) were unique. The data were examined for multicollinearity by determining tolerance and VIF. The small tolerance values of between .56 and .73 were above the recommended minimum value of .20 indicated the absence of multicollinearity (Menard, 1995).

Table 3: The LFUP correlation matrix (*n* = 87)

	1	2	3	4	5	6
ACV	–					
ACE	.547**	–				
	.000					
ACC	.515**	.518**	–			
	.000	.000				
ACD	.487**	.403**	.554**	–		
	.000	.000	.000			
ACS	.362**	.429**	.406**	.417**	–	
	.001	.000	.000	.000		
Score_ Total	.723**	.772**	.777**	.749**	.658**	–
	.000	.000	.000	.000	.000	

In the Model Summary depicted in Table 5, the sizes of the prediction variables are laid out. The overall correlation of the five subscales with functional understandings of proof was an adjusted R^2 of .78. Therefore, only 78% of the overall variation in functional understandings of proof was explained by these five subscales. This is a strong effect size.

Table 4: The effect size

Model Summary					
Model	**R**	**R Square**	**Adjusted R Square**	**Std. Error of the Estimate**	**Durbin-Watson**
1	.976[a]	.953	.950	3.30525	2.007

a. Predictors: (Constant), ACS, ACV, ACD, ACE, ACC; b. Dependent Variable: Score_Total

The function that best contributes to functional understandings of proof

In order to provide an explanation (model) of the data and thus determine which of the functions of proof best contributed to the prediction of learners' functional understandings of proof, the standardised regression coefficients (βs) were considered. By "standardised" it is meant that the coefficients are converted into a standard format thus allowing direct comparison (Wilson & MacLean, 2011). From the Coefficients table (as depicted in Table 6), it is noted that the explanatory function of proof has the largest influence (β=.548) than the others on functional understandings of proof in mathematics.

Table 5: Multiple regression analysis of five variables predicting learners' functional understanding of proof

Model	Unstandardised Coefficients		Standardised Coefficients			Collinearity Statistics			
	B	Std. Error	Beta	t	Sig.	Zero-order	Partial	Tolerance	VIF
1 (Constant)	15.558	1.902		8.179	.000				
AV	3.080	.513	.191	6.002	.000	.723	.560	.589	1.698
AE	5.225	.520	.319	10.041	.000	.772	.749	.593	1.687
AC	4.484	.557	.263	8.046	.000	.777	.671	.558	1.791
AD	5.166	.562	.288	9.201	.000	.749	.719	.610	1.640
AS	5.192	.660	.225	7.867	.000	.658	.663	.731	1.367

a. Dependent Variable: Score_Total

Qualitative Analysis

Learners' constructed-response items were scored using the LFUP rubric as depicted in Table 7. This rubric was developed as a guide to analyse and promote consistent coding from one constructed response to another. Participants' constructed-response items were assigned a label to the four different achievement levels: *adequate*, *hybrid*, *inadequate* or *indeterminate*. However, if there is no response or response is idiosyncratic (unrelated to functions of proof) it is labelled *indeterminate*. Values ascribed to responses were as follows: adequate (=3), hybrid (=2), inadequate (=1), or not classifiable (=0). A sample of fifty questionnaires whose constructed-response items were independently scored by the first author and an outside coder experienced in the teaching of high school geometry, were randomly selected. The coded questionnaires had an interrater reliability of 74.8%. To improve reliability, coding decisions were subsequently compared and an interrater reliability of 80.2% was reached following refinement of discrepancies in coding of constructed responses. The remaining questionnaires were scored primarily by the first author.

The results provided evidence that learners held relatively similar views of the mathematical practice, ranging from naïve to hybrid. Participants' responses to open-ended items provided additional evidence on their views of the mathematical practice. For instance, responses to what mathematicians do showed that many of them (89%) held the somewhat naïve views that a mathematician is a tutor to help with or to do test setting and marking of scripts. These responses were interpreted as evidence that preservice teachers indeed lack knowledge about the practice of mathematicians, but hold hybrid views of the various methods used in proving mathematical truths. However, the naïve responses on the proof of the proposition that the sum of two even numbers was an even number were interpreted as demonstrating concerns about difficulties these teachers have with regard to constructing a mathematical proof. The rubric depicted in Table 7 is a sample used to score the qualitative aspect of the scale.

Table 6: Scoring Rubric for Qualitative Data

Function	Adequate	Hybrid	Inadequate	Indeterminate
Systematisation *With examples, explain how the theorems you are asked to prove were dependent OR independent of one another*	A proof organises individual statements into a coherent deductive system **AND** Proof exposes the underlying logical relationships between statements	A proof organises individual statements into a coherent deductive system **AND/OR** Proof exposes the underlying logical relationships between statements	A proof organises individual statements into a coherent deductive system **AND/OR** Proof exposes the underlying logical relationships between statements.	No attempt made; Merely mention that "I don't know"; Response is irrelevant to the prompt.

The consistency between these responses and their Likert-scale items and thus answer to the third research question, "*How do learners view the functions of proof in mathematics?*" was analysed. Analysis of constructed-response items revealed that only a few participants (8%) demonstrated adequate views about the systematisation function. Noteworthy perhaps is that the majority of learners held inadequate views about the verification function of proof. With regard to the explanatory function of proof, fewer participants (18%) held adequate views about proof as a means to discover new results. However, a majority of participants' responses belonged to the hybrid category across all the five functions of proof. This result meant that for the majority of learners, proof represented a variety of functions some of which could be described as either consistent with current proof literature while some would be found to be naïve.

Conclusion

The purpose of this study was to validate the LFUP questionnaire by understanding the extent to which the independent attributes (five functions of proof) predicted learners' functional understandings of proof in mathematics. To do this, additional to the Likert-scale, open-ended components were employed to provide opportunities for in-depth investigation learners' views of the functions of proof in mathematics. This was done to understand which of the five functions of proof was the most important in predicting learners' functional understandings of proof. It was hypothesised that the verification function explained a significant amount of the variance in learners' functional understandings of proof. The results not only showed that the five functions explained a sizeable amount of the variation in learners' functional understanding of proof but also that the explanatory function of proof was the single most important subscale that explained the variance in the scores on functional understandings of proof when all the five functions of proof were considered together. These results suggested that the LFUP scale was both a reliable and valid instrument to measure learners' functional understanding of proof. In addition, learners' functional understanding of proof was characterised as hybrid.

The study reported in this paper made two contributions. First, it emphasises the importance for establishing validity each time an instrument is used with a different sample. Second, the LFUP scale provides a reliable tool to measure learners' appreciation of the concept of proof for meaningful learning. All in all, the instrument supports current research in learning, teaching, and assessment underscoring the importance of eliciting learners' preconceptions of the proof. Future studies may need to incorporate open-ended interviews to probe learners' responses in the questionnaire.

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References

Ball, D. L., Hoyles, C., Jahnke, H. N., & Movshovitz-Hadar, N. (2002). The teaching of proof. In L. I. Tatsien (Ed.), *Proceedings of the International Congress of Mathematicians* (Vol. III, pp. 907-920). Beijing: Higher Education.

Brook, A., & Stainton, R. J. (2001). *Knowledge and mind.* The MIT Press, Cambridge Mass, p. 29.

Cabassut, R., Conner, A. M., Ersoz, F. A., Furinghetti, F., Jahnke, H. N., & Morselli, F. (2012). Conceptions of proof – In research and in teaching. In G. Hanna, & M. D. de Villiers (Eds.), *Proof and proving in mathematics education (pp. 169-190). New York:* (pp. 169-190). New York, NY: Springer.

CadwalladerOlsker, T. (2011). What do we mean by mathematical proof? *Journal of Humanistic Mathematics, 1*(1), 33-60.

Common Core State Standards Initiative [CCSSI]. (2010). *Common Core State Standards for Mathematics (CCSSM).* Washington, DC: National Governors Association Center for Best Practices and Council of Chief State School Officers. Retrieved February 24, 2018, from http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf

de Villiers, M. D. (1990). The role and function of proof in mathematics. *Pythagoras, 24*, 17-24.

de Villiers, M. D. (2012). *Rethinking proof with the Geometer's Sketchpad* (Vol. 5). Emeryville, CA: Key Curriculum Press.

de Villiers, M. D., & Heideman, N. (2014). Conjecturing, refuting and proving within the context of dynamic geometry. *Learning and Teaching Mathematics, 17*, 20-26.

Demidov, S. S. (1980). On axiomatic and genetic construction of mathematical theories. In J. Hintikka, D. Gruender, & E. Agazzi (Ed.), *Proceedings of the 1978 Pisa Conference on the history and philosophy of science. I*, pp. 215-221. Reidel Publishing Company.

Department of Basic Education [DBE]. (2009). *The Dinaledi Schools Project: Report from a strategic engagement between the national department of education and business on increasing support for mathematics and science in education in schools.* Pretoria: Department of Basic Education.

Department of Basic Education [DBE]. (2011). *Curriculum and Assessment Policy Statement: Mathematics (Grades 10-12).* Pretoria: Department of Basic Education.

Department of Basic Education [DBE]. (2018). *National Diagnostic Report on Learner-Performance (Part 1).* Pretoria: Department of Basic Education.

Doane, D. P., & Seward, L. E. (2011). Measuring skewness. *Journal of Statistics Education, 19*(2), 1-18.

Easdown, D. (2012). The role of proof in mathematics teaching and the Plateau Principle. *Proceedings of The Australian Conference on Science and Mathematics Education (formerly UniServe Science Conference)*, (pp. 28-33).

Edwards, L. D. (1997). Exploring the territory before proof: Students' generalizations in a computer microworld for transformation geometry. *International Journal of Computers for Mathematical Learning, 2*, 187-215.

Epp, S. S. (2003). The role of logic in teaching proof. *The American Mathematical Monthly, 110*(10), 886-899.

Epstein, D., & Levy, S. (1995). Experimentation and proof in mathematics. *Notices of the AMS, 42*(6), 670-674.

Griffiths, P. A. (2000). Mathematics at the turn of the millennium. *American Mathematical Monthly, 107*(1), 1-14.

Hadas, N., Hershkowitz, R., & Schwarz, B. B. (2000). The role of contradiction and uncertainty in promoting the need to prove in dynamic geometry environments. *Educational Studies in Mathematics, 44*, 127-150.

Hanna, G., de Villiers, M., Arzare, F., Dreyfus, T., Durand-Guerrier, V., Jahnke, H. N., . . . Yevdokimov, O. (2009). Proof and proving in mathematics education: Discussion document. In F.-L. Lin, F.-J. Hsieh, & M. de Villiers (Eds.), *Proceedings of the ICMI Study 19 Conference: Proof and proving in mathematics education. National Taiwan Normal University in Taipei, Taiwan, May 10 to May 15* (pp. xix-xxx).

Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 805-842). Greenwich, CT: Information Age Publishing.

Healy, L., & Hoyles, C. (1999). Students' performance in proving: Competence or curriculum? In I. Schwank (Ed.), *European research in mathematics education I* (pp. 153-167). Osnabrueck, Germany: Forschungsinstitut für Mathematikdidaktik.

Herbst, P. G., & Miyakawa, T. (2008). When, how, and why prove theorems? A methodology for studying the perspective of geometry teachers. *ZDM Mathematics Education, 40*, 469-486.

Lockhart, P. (2002). *A mathematicians' lament.* Retrieved December 17, 2016, from https://www.maa.org/external_archive/devlin/LockhartsLament.pdf

McDowell, I. (2006). *Measuring health: A guide to rating scales and questionnaires* (3rd ed.). New York, NY: Oxford University Press.

McMillan, J. H., & Schumacher, S. (2010). *Research in education: Evidence-based inquiry* (7th ed.). Upper Saddle River, New Jersey: Pearson Education, Inc.

Marrades, R, & Gutiérrez. Proofs produced by secondary school students learning geometry in a dynamic computer environment. *Educational Studies in Mathematics, 20*(1-2), 87-125.

Menard, S. (1995). *Applied logistic regression analysis: Sage university series on quantitative applicatios in the social sciences.* Thousand Oaks, CA: Sage.

Mudaly, V. (2007). Proof and proving in secondary school. *Pythagoras, 66*, 64-75.

Myers, R. H. (1990). *Classical and modern regression application* (2nd ed.). Boston, MA: PWS-KENT.

National Council of Teachers of Mathematics [NCTM]. (2000). *Principles and Standards for School Mathematics.* Reston, VA: NCTM.

Nicolas, C. A., & Emata, C. Y. (2018). An integrative approach through reading comprehension to enhance problem solving skills of Grade 7 mathematics students. *International Journal of Innovation in Science and Mathematics Education, 26*(3), 40-64.

Rav, Y. (1999). Why do we prove theorems? *Philosophia Mathematica 7*(1), 5-41.

Reid, D. A., & Knipping, C. (2010). *Proof in mathematics education: Research, learning and teaching.* Rotterdam, The Netherlands: Sense Publishers.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 334-370). New York, NY: MacMillan.

Shongwe, B., & Mudaly, V. (2017). The development and evaluation of an instrument to measure functional understandings of proof. PONTE, *73*(12), 87-104.

Stylianides, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education, 38*(3), 289-321.

Stylianides, G. J. (2009). Reasoning-and-proving in school mathematics textbooks. *Mathematical Thinking and Learning, 11*(4), 258-288.

Stylianides, A. J., & Stylianides, G. J. (2018). Addressing key and persistent problems of students' learning: The case of proof. In A. J. Stylianides, & G. Harel (Eds.), *Advances in mathematics education research on proof and proving* (pp. 99-113). Cham, Sveits: Springer.

Tabachnick, B. G., & Fidell, L. S. (2013). *Using multivariate statistics* (6th ed.). Boston: Pearson.

Tall, D. (1989). The nature of mathematical proof. *Mathematics Teaching, 127*, 28-32.

van Hiele, P. M. (1986). *Structure and insight.* Orlando, FL: Academic Press.

Watson, A. (2008). School mathematics as a special kind of mathematics. *28*(3), 3-7.

Wilson, S., & MacLean, R. (2011). *Research methods and data analysis for psychology.* Berkshire: McGaw-Hill.